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ENERGY β-CONFORMAL CHANGE IN FINSLER GEOMETRY

A. SOLEIMAN

Department of Mathematics, Faculty of Science
Benha University, Benha, Egypt
amrsoleiman@yahoo.com

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Dedicated to the memory of Prof. Dr. A. Tamim

The present paper deals with an *intrinsic* generalization of the conformal change and energy β-change on a Finsler manifold $(M.L.)$, namely the energy β-conformal change $(\widetilde{L}^2(x,y) = e^{2\sigma(x)}L^2(x,y) + B^2(x,y)$ with $B := g(\overline{\zeta}, \overline{\eta})$; $\overline{\zeta}$ being a concurrent π-vector field and $\sigma(x)$ is a function on M). The relation between the two Barthel connections Γ and $\widetilde{\Gamma}$, corresponding to this change, is found. This relation, together with the fact that the Cartan and the Barthel connections have the same horizontal and vertical projectors, enable us to study the energy β-conformal change of the fundamental linear connection in Finsler geometry: the Cartan connection, the Berwald connection, the Chern connection and the Hashiguchi connection. Moreover, the change of their curvature tensors is obtained.

It should be pointed out that the present work is formulated in a prospective modern coordinate-free form.

Keywords: Energy β-conformal change; concurrent π-vector field; canonical spray; Barthel connection; Cartan connection; Berwald connection; Chern connection; Hashiguchi connection.

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0. Introduction

An important aim of Finsler geometry is the construction of a natural geometric framework of variational calculus and the creation of geometric models that are appropriate for dealing with different physical theories, such as general relativity, relativistic optics, particle physics and others. As opposed to Riemannian geometry, the extra degrees of freedom offered by Finsler geometry, due to the dependence of its geometric objects on the directional arguments, make this geometry potentially more suitable for dealing with such physical theories at a deeper level.

Studying Finsler geometry, however, one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of

Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

In Riemannian geometry, there is a canonical linear connection on the manifold M, whereas in Finsler geometry there is a corresponding canonical linear connection due to Cartan. However, this is not a connection on M but is a connection on $T(\mathcal{T}M)$, the tangent bundle of $\mathcal{T}M$, or on $\pi^{-1}(TM)$, the pullback of the tangent bundle TM by $\pi : \mathcal{T}M \to M$.

The infinitesimal transformations (changes) in Finsler geometry are important, not only in differential geometry, but also in application to other branches of science, especially in the process of geometrization of physical theories [10].

The theory of conformal changes in Riemannian geometry has been deeply studied (*locally and intrinsically*) by many authors. As regards to Finsler geometry, an almost complete *local* theory of conformal and β-changes has been established ([7, 8, 10, 12] etc.). On the other hand, the conformal change investigated intrinsically in Finsler geometry by [17]. The energy β-change under a concurrent π-vector field is investigated intrinsically by [18]. Very recently, [20] studied intrinsically an energy β-change under a parallel π-vector field in Finsler geometry.

The present paper is a generalization of [17, 18] where we investigate intrinsically a particular β-change, which will be referred to as an energy β-conformal change:

$$\widetilde{L}^2(x,y) = e^{2\sigma(x)}L^2(x,y) + B^2(x,y),$$

where (M, L) is a Finsler manifold admitting a concurrent π-vector field $\overline{\zeta}$, $B := g(\overline{\zeta}, \overline{\eta})$; $\overline{\eta}$ being the fundamental π-vector field and $\sigma(x)$ is a function on M. Moreover, the relation between the two Barthel connections Γ and $\widetilde{\Gamma}$, corresponding to this change, is obtained. This relation, together with the fact that the Cartan and the Barthel connections have the same horizontal and vertical projectors, enable us to study the energy β-change of the fundamental linear connections on the pullback bundle of a Finsler manifold, namely, the Cartan connection, the Berwald connection, the Chern connection and the Hashiguchi connection. Moreover, the change of their curvature tensors is concluded.

Finally, it should be pointed out that a global formulation of different aspects of Finsler geometry may give more insight into the infrastructure of physical theories and helps better understand the essence of such theories without being trapped into the complications of indices. This is one of the motivations of the present work, where all results obtained are formulated in a prospective modern coordinate-free form.

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1–3, 11, 13, 16]. We assume, unless otherwise stated, that all geometric objects treated are of class C^∞. The following notation will be used throughout this paper:

M: a real paracompact differentiable manifold of finite dimension n and of class C^∞,
$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M,
$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M,
$\pi_M : TM \to M$: the tangent bundle of M,
$\pi^*_M : T^*M \to M$: the cotangent bundle of M,
$\pi : \mathcal{T}M \to M$: the subbundle of nonzero vectors tangent to M,
$V(TM)$: the vertical subbundle of the bundle TTM,
$P : \pi^{-1}(TM) \to \mathcal{T}M$: the pullback of the tangent bundle TM by π,
$P^* : \pi^{-1}(T^*M) \to \mathcal{T}M$: the pullback of the cotangent bundle T^*M by π,
$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of $\pi^{-1}(TM)$,
$\mathfrak{X}^*(\pi(M))$: the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of $\pi^{-1}(T^*M)$,
i_X: the interior product with respect to $X \in \mathfrak{X}(M)$,
df: the exterior derivative of $f \in \mathfrak{F}(M)$,
$d_L := [i_L, d]$, i_L: being the interior derivative with respect to a vector form L.

Elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(\mathcal{T}M)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \to \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \to 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) := j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \to T_u(T_{\pi_M(v)}M)$. The vector 1-form J on TM defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM. The vertical vector field $\mathcal{C}$ on TM defined by $\mathcal{C} := \gamma \circ \overline{\eta}$ is called the fundamental or the canonical (Liouville) vector field.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map

$$K : T\mathcal{T}M \to \pi^{-1}(TM) : X \mapsto D_X\overline{\eta},$$

called the connection (or the deflection) map of D. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space to M at u. The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M), \quad \forall\, u \in \mathcal{T}M. \tag{1.1}$$

If M is endowed with a regular connection, then the vector bundle maps

$$\gamma : \pi^{-1}(TM) \to V(\mathcal{T}M),$$

$$\rho|_{H(\mathcal{T}M)} : H(\mathcal{T}M) \to \pi^{-1}(TM),$$

$$K|_{V(\mathcal{T}M)} : V(\mathcal{T}M) \to \pi^{-1}(TM),$$

are vector bundle isomorphisms. Let us denote $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, then

$$\rho \circ \beta = \mathrm{id}_{\pi^{-1}(TM)}, \quad \beta \circ \rho = \begin{cases} \mathrm{id}_{H(\mathcal{T}M)} & \text{on } H(\mathcal{T}M), \\ 0 & \text{on } V(\mathcal{T}M). \end{cases} \tag{1.2}$$

The map β will be called the horizontal map of the connection D.

According to the direct sum decomposition (1.1), a regular connection D gives rise to a horizontal projector h_D and a vertical projector v_D, given by

$$h_D = \beta \circ \rho, \quad v_D = I - \beta \circ \rho, \tag{1.3}$$

where I is the identity endomorphism on $T(TM)$: $I = \mathrm{id}_{T(TM)}$.

The (classical) torsion tensor $\mathbf{T}$ of the connection D is defined by,

$$\mathbf{T}(X, Y) = D_X \rho Y - D_Y \rho X - \rho[X, Y] \quad \forall\, X, Y \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors, denoted by Q and T, respectively, are defined by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}), \quad T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}) \quad \forall\, \overline{X},\ \overline{Y} \in \mathfrak{X}(\pi(M)).$$

The (classical) curvature tensor $\mathbf{K}$ of the connection D is defined by,

$$\mathbf{K}(X, Y)\rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X,Y]}\rho Z \quad \forall\, X, Y, Z \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by R, P and S respectively, are defined by,

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z},$$

$$S(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

The contracted curvature tensors, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ respectively, are also known as the (v)h-, (v)hv- and (v)v-torsion tensors and are defined by,

$$\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y})\overline{\eta}.$$

On a Finsler manifold (M, L), there are *canonically* associated four linear connections on $\pi^{-1}(TM)$ [19]: the Cartan connection, the Chern (Rund) connection, the Hashiguchi connection and the Berwald connection. Each of these connections is regular with (h)hv-torsion T satisfying $T(\overline{X}, \overline{\eta}) = 0$. The following theorem guarantees the existence and uniqueness of the Cartan connection on the pullback bundle.

Theorem 1.1 ([15]). *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such* *that*

(a) ∇ *is metric*: $\nabla g = 0$,
(b) *The* $(h)h$*-torsion of* ∇ *vanishes*: $Q = 0$,
(c) *The* $(h)hv$*-torsion* T *of* ∇ *satisfies*: $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.

Such a connection is called the Cartan connection associated with the Finsler manifold (M, L).

Concerning the Berwald connection on the pullback bundle, we have the following theorem.

Theorem 1.2 ([15]). *Let* (M, L) *be a Finsler manifold. There exists a unique regular connection* D° *on* $\pi^{-1}(TM)$ *such that*

(a) $D^\circ_{h^\circ X} L = 0$,
(b) D° *is torsion-free*: $\boldsymbol{T}^\circ = 0$,
(c) *The* $(v)hv$*-torsion tensor* $\widehat{P^\circ}$ *of* D° *vanishes*: $\widehat{P^\circ}(\overline{X}, \overline{Y}) = 0$.

Such a connection is called the Berwald connection associated with the Finsler manifold (M, L).

Theorem 1.3 ([15]). *Let* (M, L) *be a Finsler manifold. The Berwald connection* D° *is expressed in terms of the Cartan connection* ∇ *as*

$$D^\circ_X \overline{Y} = \nabla_X \overline{Y} + \widehat{P}(\rho X, \overline{Y}) - T(KX, \overline{Y}), \quad \forall\, X \in \mathfrak{X}(TM), \quad \overline{Y} \in \mathfrak{X}(\pi(M)).$$

In particular, we have:

(a) $D^\circ_{\gamma\overline{X}} \overline{Y} = \nabla_{\gamma\overline{X}} \overline{Y} - T(\overline{X}, \overline{Y})$,
(b) $D^\circ_{\beta\overline{X}} \overline{Y} = \nabla_{\beta\overline{X}} \overline{Y} + \widehat{P}(\overline{X}, \overline{Y})$.

We terminate this section by some concepts and results concerning the Klein–Grifone approach to intrinsic Finsler geometry. For more details, we refer to [5, 6, 9].

A semispray is a vector field X on TM, C^∞ on $\mathcal{T}M$, C^1 on TM, such that $\rho \circ X = \overline{\eta}$. A semispray X which is homogeneous of degree 2 in the directional argument ($[\mathcal{C}, X] = X$) is called a spray.

Proposition 1.4 ([9]). *Let* (M, L) *be a Finsler manifold. The vector field* G *on* TM *defined by* $i_G\Omega = -dE$ *is a spray, where* $E := \frac{1}{2}L^2$ *is the energy function and* $\Omega := dd_JE$. *Such a spray is called the canonical spray.*

A nonlinear connection on M is a vector 1-form Γ on TM, C^∞ on $\mathcal{T}M$, C^0 on TM, such that

$$J\Gamma = J, \quad \Gamma J = -J.$$

The horizontal and vertical projectors h_Γ and v_Γ associated with Γ are defined by $h_\Gamma := \frac{1}{2}(I + \Gamma)$ and $v_\Gamma := \frac{1}{2}(I - \Gamma)$. To each nonlinear connection Γ there is associated a semispray S defined by $S = h_\Gamma S'$, where S' is an arbitrary semispray.

A nonlinear connection Γ is homogeneous if $[\mathcal{C},\Gamma]=0$. The torsion of a nonlinear connection Γ is the vector 2-form t on TM defined by $t:=\frac{1}{2}[J,\Gamma]$. The curvature of Γ is the vector 2-form $\mathfrak{R}$ on TM defined by $\mathfrak{R}:=-\frac{1}{2}[h_\Gamma,h_\Gamma]$. A nonlinear connection Γ is said to be conservative if $d_{h_\Gamma}E=0$.

Theorem 1.5 ([6]). *On a Finsler manifold (M,L), there exists a unique conservative homogeneous nonlinear connection with zero torsion. It is given by*:

$$\Gamma=[J,G],$$

where G is the canonical spray.

Such a nonlinear connection is called the canonical connection, the Barthel connection or the Cartan nonlinear connection associated with (M,L).

It should be noted that the semispray associated with the Barthel connection is a spray, which is the canonical spray.

Proposition 1.6 ([17]). *Under a change $L\to\widetilde{L}$ of Finsler structures on M, the corresponding Barthel connections Γ and $\widetilde{\Gamma}$ are related by*

$$\widetilde{\Gamma}=\Gamma-2\L,\quad \textit{with } \L:=\gamma o N o\rho. \tag{1.4}$$

Moreover, we have $\widetilde{h}=h-\L$, $\widetilde{v}=v+\L$ or $\widetilde{\beta}=\beta-\gamma\circ N$, $\widetilde{K}=K+N\circ\rho$ (the definition of N is found in [17]*).*

2. Energy β-Conformal Change and Barthel Connection

In the present and the next sections we consider a perturbation, by a concurrent π-vector field $\overline{\zeta}(x)$ and a positive differentiable function $\sigma(x)$ on M, of the energy function $E=\frac{1}{2}L^2$ of a Finsler structure L.

Let (M,L) be a Finsler manifold. Consider the change

$$\widetilde{L}^2(x,y)=e^{2\sigma(x)}L^2(x,y)+B^2(x,y),\quad \text{with } B:=g(\overline{\eta},\overline{\zeta})=:\alpha(\overline{\eta}), \tag{2.1}$$

$\widetilde{L}$ defines a new Finsler structure on M. The Finsler structure $\widetilde{L}$ is said to be obtained from the Finsler structure L by the β-conformal change (2.1). The β-conformal change (2.1) will be referred to as an *energy β-conformal change* (as it can be written in the form $\widetilde{E}=e^{2\sigma(x)}E+\frac{1}{2}B^2$, where E and $\widetilde{E}$ are the energy functions corresponding to the Lagrangians L and $\widetilde{L}$, respectively).

The following definition and three lemmas are useful for subsequence use.

Definition 2.1 ([18]). Let (M,L) be a Finsler manifold. A π-vector field $\overline{\zeta}\in\mathfrak{X}(\pi(M))$ is called a concurrent π-vector field if it satisfies the following conditions:

$$\nabla_{\beta\overline{X}}\overline{\zeta}=-\overline{X},\quad \nabla_{\gamma\overline{X}}\overline{\zeta}=0. \tag{2.2}$$

In other words, $\overline{\zeta}$ is a concurrent π-vector field if $\nabla_X\overline{\zeta}=-\rho X$ for all $X\in\mathfrak{X}(\mathcal{T}M)$, or briefly, $\nabla\overline{\zeta}=-\rho$.

Lemma 2.2 ([18]). *A concurrent π-vector field $\overline{\zeta}$ and its associated π-form $\alpha:=i_{\overline{\zeta}}g$ are independent of the directional argument y.*

Lemma 2.3. *Under energy β-conformal change* (2.1), *we have*

(a) $(d_J e^{2\sigma(x)} E)(X) = e^{2\sigma(x)} g(\rho X, \overline{\eta})$.
(b) $(d\sigma \wedge d_J E)(\gamma \overline{X}, Y) = 0$.
(c) $dd_J B^2(\gamma \overline{X}, \beta \overline{Y}) = 2\alpha(\overline{X})\alpha(\overline{Y})$, $dd_J B^2(\gamma \overline{X}, \gamma \overline{Y}) = 0$.
(d) $dd_J B^2(\beta \overline{X}, \beta \overline{Y}) = 2(\alpha \wedge i_{\overline{\eta}} g)(\overline{X}, \overline{Y})$.

Proof. (a) From $g(\overline{\eta}, \overline{\eta}) = 2E^2$ and $\nabla g = 0$, one can show that

$$d_J E(X) = g(\rho X, \overline{\eta}), \quad \forall\, X \in \mathfrak{X}(\mathcal{T}M).$$

From which, together with the fact that $\sigma(x)$ is independent of y, the result follows.

(b) Follows from the two identities $d\sigma(\gamma \overline{X}) = 0$ and $d_J E(\gamma \overline{X}) = 0$.

(c) Making use of Lemma 2.2 and Definition 2.1, we have

$$\begin{aligned} &dd_J B^2(\gamma \overline{X}, \beta \overline{Y}) \\ &\quad = \gamma \overline{X} \cdot d_J B^2(\beta \overline{Y}) - \beta \overline{Y} \cdot d_J B^2(\gamma \overline{X}) - d_J B^2([\gamma \overline{X}, \beta \overline{Y}]) \\ &\quad = 2\gamma \overline{X} \cdot (B\gamma \overline{Y} \cdot B) - 2\beta \overline{Y} \cdot (BJ\gamma \overline{X} \cdot B) - 2BJ[\gamma \overline{X}, \beta \overline{Y}] \cdot B \\ &\quad = 2\gamma \overline{X} \cdot (Bg(\overline{Y}, \overline{\zeta})) - 2Bg(\rho[\gamma \overline{X}, \beta \overline{Y}], \overline{\zeta}) \\ &\quad = 2\left\{g(\overline{X}, \overline{\zeta})g(\overline{Y}, \overline{\zeta}) + B\gamma \overline{X} \cdot g(\overline{Y}, \overline{\zeta})\right\} - 2Bg(\nabla_{\gamma \overline{X}} \overline{Y} - T(\overline{X}, \overline{Y}), \overline{\zeta}). \end{aligned}$$

Now, noting that $T(\overline{X}, \overline{\zeta})$ ([18, Corollary 3.5]), the above relation reduces to

$$\begin{aligned} dd_J B^2(\gamma \overline{X}, \beta \overline{Y}) &= 2\{g(\overline{X}, \overline{\zeta})g(\overline{Y}, \overline{\zeta}) + Bg(\nabla_{\gamma \overline{X}} \overline{Y}, \overline{\zeta})\} - 2Bg(\nabla_{\gamma \overline{X}} \overline{Y}, \overline{\zeta}) \\ &= 2\alpha(\overline{X})\alpha(\overline{Y}). \end{aligned}$$

Similarly, $dd_J B^2(\gamma \overline{X}, \gamma \overline{Y}) = 0$.

(d) The proof is analogous to that of (b). □

Lemma 2.4. *Let* (M, L) *be a Finsler manifold. Let* g *be the Finsler metric associated with* L *and* ∇ *the Cartan connection on* $\pi^{-1}(TM)$*. Then, the following relation holds*:

$$g(\overline{X}, \overline{Y}) = \Omega(\gamma \overline{X}, \beta \overline{Y}) \quad \textit{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)), \tag{2.3}$$

where $\Omega := dd_J E$ *and* β *is the connection map associated with* ∇.

Proof. Using the relations $d_J E(X) = g(\rho X, \overline{\eta})$, $\nabla g = 0$, $T(\rho X, \overline{\eta}) = 0$ and $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$, we get

$$\begin{aligned} \Omega(\gamma \overline{X}, \beta \overline{Y}) &= \gamma \overline{X} \cdot d_J E(\beta \overline{Y}) - \beta \overline{Y} \cdot d_J E(\gamma \overline{X}) - d_J E([\gamma \overline{X}, \beta \overline{Y}]) \\ &= \gamma \overline{X} \cdot g(\overline{Y}, \overline{\eta}) - g(\rho[\gamma \overline{X}, \beta \overline{Y}], \overline{\eta}) \end{aligned}$$

$$= g(\nabla_{\gamma\overline{X}}\overline{Y}, \overline{\eta}) + g(\overline{Y}, \nabla_{\gamma\overline{X}}\overline{\eta}) - g(\rho[\gamma\overline{X}, \beta\overline{Y}], \overline{\eta})$$
$$= g(\overline{X}, \overline{Y}) + g(T(\overline{X}, \overline{Y}), \overline{\eta}) = g(\overline{X}, \overline{Y}),$$

which proves the required relation. □

The following result gives the relationship between g and $\widetilde{g}$.

Proposition 2.5. *Under the energy β-conformal change* (2.1), *the Finsler metrics g and $\widetilde{g}$ are related by*

$$\widetilde{g}(\overline{X}, \overline{Y}) = e^{2\sigma(x)} g(\overline{X}, \overline{Y}) + \alpha(\overline{X})\alpha(\overline{Y}), \quad \textit{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)), \tag{2.4}$$

α being the π-form associated with $\overline{\zeta}$ under the duality defined by the metric g.

Proof. The proof follows by applying the operator $\frac{1}{2} dd_J$ on both sides of (2.1), taking into account Lemmas 2.4 and 2.3.

In more details,

$$\widetilde{g}(\overline{X}, \overline{Y}) = dd_J\widetilde{E}(\gamma\overline{X}, \widetilde{\beta}\overline{Y}) = dd_J\widetilde{E}(\gamma\overline{X}, \beta\overline{Y})$$
$$= e^{2\sigma(x)} dd_J E(\gamma\overline{X}, \beta\overline{Y}) + 2e^{2\sigma(x)}(d\sigma \wedge d_J E)(\gamma\overline{X}, \beta\overline{Y}) + \frac{1}{2} dd_J B^2(\gamma\overline{X}, \beta\overline{Y}),$$
$$= e^{2\sigma(x)} g(\overline{X}, \overline{Y}) + \alpha(\overline{X})\alpha(\overline{Y}).$$
□

Corollary 2.6. *Under the energy β-conformal change* (2.1), *the exterior 2-forms Ω and $\widetilde{\Omega}$ are related by*:

$$\widetilde{\Omega} = e^{2\sigma(x)}\Omega + 2e^{2\sigma(x)}(d\sigma \wedge d_J E) + \frac{1}{2} dd_J B^2.$$

Theorem 2.7. *Let (M, L) and $(M, \widetilde{L})$ be two Finsler manifolds related by the energy β-conformal change* (2.1). *The associated canonical sprays G and $\widetilde{G}$ are related by*:

$$\widetilde{G} = G + 2\{E\gamma\overline{\sigma} - \sigma_1\mathcal{C}\} + \mathcal{A}\gamma\overline{\zeta},$$

where

$$\left.\begin{aligned} g(\rho X, \overline{\sigma}) &:= d\sigma(X) = d\sigma(hX), \\ \sigma_1 &:= d\sigma(G), \\ \mathcal{A} &:= \frac{L^2 + 2\sigma_1 B - 2\sigma_2 E}{e^{2\sigma(x)} + p^2}, \\ p^2 &:= g(\overline{\zeta}, \overline{\zeta}), \\ \sigma_2 &:= d\sigma(\beta\overline{\zeta}). \end{aligned}\right\} \tag{2.5}$$

Proof. From the above corollary, we have

$$\widetilde{\Omega} = e^{2\sigma(x)}\Omega + 2e^{2\sigma(x)}(d\sigma \wedge d_J E) + \frac{1}{2}\, dd_J B^2.$$

As the difference between two sprays is a vertical vector field, assume that $\widetilde{G} = G + \gamma\overline{\mu}$, for some $\overline{\mu} \in \mathfrak{X}(\pi(M))$, then we have

$$\begin{aligned}
-d\widetilde{E}(X) &= i_{\widetilde{G}}\, \widetilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\, \widetilde{\Omega}(X) \\
&= e^{2\sigma(x)} i_G\, \Omega(X) + 2e^{2\sigma(x)} i_G(d\sigma \wedge d_J E)(X) + \frac{1}{2} i_G\, dd_J B^2(X) \\
&\quad + e^{2\sigma(x)} i_{\gamma\overline{\mu}}\Omega(X) + 2e^{2\sigma(x)} i_{\gamma\overline{\mu}}(d\sigma \wedge d_J E) + \frac{1}{2} i_{\gamma\overline{\mu}} dd_J B^2(X).
\end{aligned} \tag{2.6}$$

Now, we compute the terms on the right-hand side (using Lemmas 2.3 and 2.4 and Proposition 1.4):

$$\begin{aligned}
i_G\Omega(X) &= -dE(X), \\
i_{\gamma\overline{\mu}}\Omega(X) &= \Omega(\gamma\overline{\mu}, X) = \Omega(\gamma\overline{\mu}, \gamma KX + \beta\rho X) = \Omega(\gamma\overline{\mu}, \beta\rho X) = g(\overline{\mu}, \rho X), \\
i_G(d\sigma \wedge d_J E)(X) &= (d\sigma \wedge d_J E)(G, \beta\rho X) + (d\sigma \wedge d_J E)(G, \gamma KX) \\
&= d\sigma(G) g(\rho X, \overline{\eta}) - 2E d\sigma(X), \\
i_{\gamma\overline{\mu}}(d\sigma \wedge d_J E)(X) &= (d\sigma \wedge d_J E)(\gamma\overline{\mu}, \beta\rho X) + (d\sigma \wedge d_J E)(\gamma\overline{\mu}, \gamma KX) = 0, \\
\frac{1}{2} i_G dd_J B^2(X) &= \frac{1}{2}\{dd_J B^2(\beta\overline{\eta}, \beta\rho X) - dd_J B^2(\gamma KX, \beta\overline{\eta})\} \\
&= \{g(\overline{\eta}, \overline{\zeta}) g(\rho X, \overline{\eta}) - g(\overline{\eta}, \overline{\eta}) g(\rho X, \overline{\zeta})\} - g(KX, \overline{\zeta}) g(\overline{\eta}, \overline{\zeta}) \\
&= \{g(\overline{\eta}, \overline{\zeta}) g(\rho X, \overline{\eta}) - g(\overline{\eta}, \overline{\eta}) g(\rho X, \overline{\zeta})\} \\
&\quad - \{X \cdot g(\overline{\eta}, \overline{\zeta}) + g(\overline{\eta}, \rho X)\} g(\overline{\eta}, \overline{\zeta}) \\
&= -L^2 g(\overline{\zeta}, \rho X) - B dB(X), \\
\frac{1}{2} i_{\gamma\overline{\mu}} dd_J B^2(X) &= \frac{1}{2} dd_J B^2(\gamma\overline{\mu}, \beta\rho X + \gamma KX) = \frac{1}{2} dd_J B^2(\gamma\overline{\mu}, \beta\rho X) \\
&= g(\overline{\zeta}, \overline{\mu}) g(\overline{\zeta}, \rho X).
\end{aligned}$$

From these, together with

$$d\widetilde{E}(X) = e^{2\sigma(x)} dE(X) + 2e^{2\sigma(x)} E d\sigma(X) + B dB(X),$$

Eq. (2.6) reduces to

$$\begin{aligned}
&e^{2\sigma(x)} g(\overline{\mu}, \rho X) - L^2 g(\overline{\zeta}, \rho X) + g(\overline{\zeta}, \overline{\mu}) g(\overline{\zeta}, \rho X) \\
&\quad + 2e^{2\sigma(x)} d\sigma(G) g(\overline{\eta}, \rho X) - 2e^{2\sigma(x)} E d\sigma(X) = 0.
\end{aligned} \tag{2.7}$$

Setting $X = \beta\overline{\zeta}$, taking into account (2.5), we obtain

$$g(\overline{\zeta}, \overline{\mu}) = \frac{L^2 p^2 - 2e^{2\sigma(x)}\sigma_1 B + 2e^{2\sigma(x)}\sigma_2 E}{e^{2\sigma(x)} + p^2}.$$

From which, together with (2.7) and (2.5), we obtain

$$\overline{\mu} = 2\{E\bar{\sigma} - \sigma_1\overline{\eta}\} + \mathcal{A}\overline{\zeta}.$$

Hence the result. □

Theorem 2.8. *Let (M, L) and $(M, \widetilde{L})$ be two Finsler manifolds related by the energy β-conformal change* (2.1). *The associated Barthel connections Γ and $\widetilde{\Gamma}$ are related by*

$$\widetilde{\Gamma} = \Gamma - 2\{E[\gamma\overline{\sigma}, J] - d_J E \otimes \gamma\overline{\sigma} + \sigma_1 J + d\sigma \otimes \mathcal{C}\} + \frac{2\{(1-\sigma_2)d_J E + Bd\sigma + \sigma_1 d_J B\}}{e^{2\sigma(x)} + p^2} \otimes \gamma\overline{\zeta}\,.$$

Proof. From Theorem 2.7 and the formula [4]

$$[fX, J] = f[X, J] + df \wedge i_X J - d_J f \otimes X,$$

we obtain

$$\begin{aligned}
\widetilde{\Gamma} &= [J, \widetilde{G}] = [J, G + 2\{E\gamma\overline{\sigma} - \sigma_1\mathcal{C}\} + \mathcal{A}\,\gamma\overline{\zeta}] \\
&= [J, G] - 2[E\gamma\overline{\sigma} - \sigma_1\mathcal{C}, J] - [\mathcal{A}\gamma\overline{\zeta}, J] \\
&= \Gamma - 2\{E[\gamma\overline{\sigma}, J] + dE \wedge i_{\gamma\overline{\sigma}}J - d_J E \otimes \gamma\overline{\sigma}\} \\
&\quad + 2\{\sigma_1[\mathcal{C}, J] + d\sigma_1 \wedge i_{\mathcal{C}}J - d_J\sigma_1 \otimes \mathcal{C}\} \\
&\quad - \left\{\mathcal{A}[\gamma\overline{\zeta}, J] + d\mathcal{A} \wedge i_{\gamma\overline{\zeta}}J - d_J\mathcal{A} \otimes \gamma\overline{\zeta}\right\}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&[\mathcal{C}, J] = -J, \qquad i_{\mathcal{C}}J = 0, \\
&i_{\gamma\overline{\sigma}}J = 0, \qquad i_{\gamma\overline{\zeta}}J = 0, \\
&d_J\sigma_1 = d\sigma, \quad d_J\sigma_2 = 0, \\
&d_J\mathcal{A} = \frac{2\{(1-\sigma_2)d_J E + Bd\sigma + \sigma_1 d_J B\}}{e^{2\sigma(x)} + p^2},
\end{aligned}$$

whereas,

$$\begin{aligned}
[\gamma\overline{\zeta}, J]X &= [\gamma\overline{\zeta}, JX] - J[\gamma\overline{\zeta}, X] \\
&= \gamma\{\nabla_{\gamma\overline{\zeta}}\rho X - \nabla_{JX}\overline{\zeta}\} - \gamma\{\nabla_{\gamma\overline{\zeta}}\,\rho X - T(\overline{\zeta}, \rho X)\} = 0,
\end{aligned}$$

by [16, Lemma 4.3] and the fact that $T(\overline{\zeta}, \rho X) = 0$. Consequently,

$$\widetilde{\Gamma} = \Gamma - 2\{E[\gamma\overline{\sigma}, J] - d_J E \otimes \gamma\overline{\sigma} + \sigma_1 J + d\sigma \otimes \mathcal{C}\} + d_J\mathcal{A} \otimes \gamma\overline{\zeta},$$

which proves the result. □

As a consequence of the given results, we have the following interesting special cases which retrieve here a result of [17, 18]:

Theorem 2.9. (a) *Let the energy β-conformal change* (2.1) *be a conformal change* ($\zeta = 0$), *then we have*

$$\widetilde{g}(\overline{X}, \overline{Y}) = e^{2\sigma(x)} g(\overline{X}, \overline{Y}),$$
$$\widetilde{G} = G + 2\{E\gamma\overline{\sigma} - \sigma_1 \mathcal{C}\},$$
$$\widetilde{\Gamma} = \Gamma - 2\{E[\gamma\overline{\sigma}, J] - d_J E \otimes \gamma\overline{\sigma} + \sigma_1 J + d\sigma \otimes \mathcal{C}\}.$$

(b) *Let the energy β-conformal change* (2.1) *be an energy β-change* ($\sigma = 0$), *then we have*

$$\widetilde{g}(\overline{X}, \overline{Y}) = g(\overline{X}, \overline{Y}) + \alpha(\overline{X})\alpha(\overline{Y}),$$
$$\widetilde{G} = G + \frac{L^2}{1+p^2}\gamma\overline{\zeta},$$
$$\widetilde{\Gamma} = \Gamma + 2\,\frac{d_J E}{1+p^2} \otimes \gamma\overline{\zeta}.$$

Remark 2.10. Comparing Theorem 2.8 and Proposition 1.6, we find that

$$\widetilde{\Gamma} = \Gamma - 2Ł,$$

where

$$Ł := \{E[\gamma\overline{\sigma}, J] - d_J E \otimes \gamma\overline{\sigma} + \sigma_1 J + d\sigma \otimes \mathcal{C}\} - \frac{\{(1-\sigma_2)d_J E + B d\sigma + \sigma_1 d_J B\}}{e^{2\sigma(x)} + p^2} \otimes \gamma\overline{\zeta}.$$

Theorem 2.11. *Under the energy β-change* (2.1), *the curvature tensors $\widetilde{\mathfrak{R}}$ and $\mathfrak{R}$ of the associated Barthel connections $\widetilde{\Gamma}$ and Γ are related by*

$$\widetilde{\mathfrak{R}}(X,Y) = \mathfrak{R}(X,Y) - [ŁX, ŁY] - Ł[hX, hY] + \mathfrak{A}_{X,Y}\{v[hX, ŁY] + Ł[hX, ŁY]\}, \tag{2.8}$$

where $\mathfrak{A}_{X,Y}\{\Theta(X,Y)\} = \Theta(X,Y) - \Theta(Y,X)$ and Ł is given by Remark 2.10.

Proof. The proof follows from the fact that $\mathfrak{R}(X,Y) = -v[hX, hY]$ [14] and $Ł[ŁX, ŁY] = 0$, taking Remark 2.10 into account. □

3. Fundamental Connections Under an Energy β-Conformal Change

In this section, we investigate the transformation of the fundamental linear connections of Finsler geometry, as well as their curvature tensors, under the energy β-conformal change (2.1).

We start our investigation with the Cartan connection.

The following lemmas are useful for subsequent use.

Lemma 3.1 ([15]). *Let (M, L) be a Finsler manifold and g the Finsler metric associated with L. The Cartan connection ∇ is completely determined by the relations*:

(a) $2g(\nabla_{vX}\rho Y, \rho Z) = vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, vX]) + g(\rho Z, \rho[vX, hY])$.

(b) $2g(\nabla_{hX}\rho Y, \rho Z) = hX \cdot g(\rho Y, \rho Z) + hY \cdot g(\rho Z, \rho X) - hZ \cdot g(\rho X, \rho Y) - g(\rho X, \rho[hY, hZ]) + g(\rho Y, \rho[hZ, hX]) + g(\rho Z, \rho[hX, hY])$.

Since the Cartan connection ∇ has the same horizontal and vertical projectors as the Barthel connection $\Gamma =: [J, G]$ [15], we get

Lemma 3.2. *Under the energy β-conformal change* (2.1), *we have*

$$\widetilde{h} = h - Ł, \quad \widetilde{v} = v + Ł, \quad \text{or equivalently} \quad \widetilde{\beta} = \beta - Ł\beta, \quad \widetilde{K} = K + KŁ,$$

where

$$\begin{aligned}\gamma o N o\rho =: Ł &:= \{E[\gamma\overline{\sigma}, J] - d_J E \otimes \gamma\overline{\sigma} + \sigma_1 J + d\sigma \otimes \mathcal{C}\} \\ &\quad - \frac{\{(1-\sigma_2) d_J E + B d\sigma + \sigma_1 d_J B\}}{e^{2\sigma(x)} + p^2} \otimes \gamma\overline{\zeta} \\ &=: L_1 + Ł_0 \otimes \gamma\overline{\zeta}.\end{aligned}$$

Theorem 3.3. *Let (M, L) and $(M, \widetilde{L})$ be two Finsler manifolds related by the energy β-conformal change* (2.1). *Then the associated Cartan connections ∇ and $\widetilde{\nabla}$ are related by*:

$$\widetilde{\nabla}_X \overline{Y} = \nabla_X \overline{Y} + \omega(X, \overline{Y}), \tag{3.1}$$

where

$$\begin{aligned}&\omega(X, \overline{Y}) \\ &\quad := (d\sigma(hX))\overline{Y} + (d\sigma(\beta\overline{Y}))\rho X - g(\rho X, \overline{Y})\overline{\sigma} - \boldsymbol{T}(Ł\beta\overline{Y}, \rho X) + T'(ŁX, \beta\overline{Y}) \\ &\qquad - \left\{\frac{d\sigma(hX) g(\overline{Y}, \overline{\zeta}) + d\sigma(\beta\overline{Y}) g(\rho X, \overline{\zeta}) - g(\overline{\sigma}, \overline{\zeta}) g(\rho X, \overline{Y}) + g(\rho X, \overline{Y})}{e^{2\sigma(x)} + p^2}\right\}\overline{\zeta},\end{aligned} \tag{3.2}$$

$\overline{\sigma}$ being a π-vector field defined by (2.5) *and T' a 2-form on TM, with values in $\pi^{-1}(TM)$, defined by*

$$g(T'(ŁX, hY), \rho Z) = g(\mathbf{T}(ŁZ, hY), \rho X).$$

In particular,

(a) $\widetilde{\nabla}_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y}$,

(b) $\widetilde{\nabla}_{\widetilde{\beta}\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + B(\overline{X}, \overline{Y}) - \nabla_{Ł\beta\overline{X}}\overline{Y}$,

where $B(\overline{X}, \overline{Y}) := \omega(\beta\overline{X}, \overline{Y})$ and $Ł$ is given by Lemma 3.2.

Proof. Using Lemma 3.1(a), Proposition 2.5 and Lemma 3.2, noting the fact that $\rho[\widetilde{hZ}, \widetilde{v}X] = \rho[hZ, \widetilde{v}X]$ (as $\rho[ⱢZ, \widetilde{v}X] = 0$), we get

$$\begin{aligned}
2\widetilde{g}(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \rho Z) &= \widetilde{v}X \cdot \widetilde{g}(\rho Y, \rho Z) + \widetilde{g}(\rho Y, \rho[hZ, \widetilde{v}X]) + \widetilde{g}(\rho Z, \rho[\widetilde{v}X, hY]) \\
&= vX \cdot \widetilde{g}(\rho Y, \rho Z) + \widetilde{g}(\rho Y, \rho[hZ, vX]) + \widetilde{g}(\rho Z, \rho[vX, hY]) \\
&\quad + ⱢX \cdot \widetilde{g}(\rho Y, \rho Z) + \widetilde{g}(\rho Y, \rho[hZ, ⱢX]) + \widetilde{g}(\rho Z, \rho[ⱢX, hY]) \\
&= vX \cdot \{e^{2\sigma(x)} g(\rho Y, \rho Z) + g(\rho Y, \overline{\zeta}) g(\rho Z, \overline{\zeta})\} \\
&\quad + e^{2\sigma(x)} g(\rho Y, \rho[hZ, vX]) + g(\rho Y, \overline{\zeta}) g(\rho[hZ, vX], \overline{\zeta}) \\
&\quad + e^{2\sigma(x)} g(\rho Z, \rho[vX, hY]) + g(\rho Z, \overline{\zeta}) g(\rho[vX, hY], \overline{\zeta}) \\
&\quad + ⱢX \cdot \{e^{2\sigma(x)} g(\rho Y, \rho Z) + g(\rho Y, \overline{\zeta}) g(\rho Z, \overline{\zeta})\} \\
&\quad + e^{2\sigma(x)} g(\rho Y, \rho[hZ, ⱢX]) + g(\rho Y, \overline{\zeta}) g(\rho[hZ, ⱢX], \overline{\zeta}) \\
&\quad + e^{2\sigma(x)} g(\rho Z, \rho[ⱢX, hY]) + g(\rho Z, \overline{\zeta}) g(\rho[ⱢX, hY], \overline{\zeta}). \\
&= e^{2\sigma(x)} \{vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, vX]) + g(\rho Z, \rho[vX, hY])\} \\
&\quad + e^{2\sigma(x)} \{ⱢX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, ⱢX]) + g(\rho Z, \rho[ⱢX, hY])\} \\
&\quad + \{g(\nabla_{vX}\rho Y, \overline{\zeta}) g(\rho Z, \overline{\zeta}) + g(\rho Y, \overline{\zeta}) g(\nabla_{vX}\rho Z, \overline{\zeta}) \\
&\quad + g(\rho Y, \overline{\zeta}) g(\mathbf{T}(vX, hZ) - \nabla_{vX}\rho Z, \overline{\zeta}) \\
&\quad + g(\rho Z, \overline{\zeta}) g(\nabla_{vX}\rho Y - \mathbf{T}(vX, hY), \overline{\zeta})\} \\
&\quad + \{g(\nabla_{ⱢX}\rho Y, \overline{\zeta}) g(\rho Z, \overline{\zeta}) + g(\rho Y, \overline{\zeta}) g(\nabla_{ⱢX}\rho Z, \overline{\zeta}) \\
&\quad + g(\rho Y, \overline{\zeta}) g(\mathbf{T}(ⱢX, hZ) - \nabla_{ⱢX}\rho Z, \overline{\zeta}) \\
&\quad + g(\rho Z, \overline{\zeta}) g(\nabla_{ⱢX}\rho Y - \mathbf{T}(ⱢX, hY), \overline{\zeta})\}.
\end{aligned}$$

As $g(\mathbf{T}(vX, hY), \overline{\zeta}) = -g(T(KX, \overline{\zeta}), \rho Y) = 0$, the above relation takes the form:

$$\begin{aligned}
2\widetilde{g}(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \rho Z) &= 2e^{2\sigma(x)} g(\nabla_{vX}\rho Y, \rho Z) + 2e^{2\sigma(x)} g(\nabla_{ⱢX}\rho Y, \rho Z) \\
&\quad + 2g(\rho Z, \overline{\zeta}) g(\nabla_{vX}\rho Y, \overline{\zeta}) + 2g(\rho Z, \overline{\zeta}) g(\nabla_{ⱢX}\rho Y, \overline{\zeta}) \\
&= 2e^{2\sigma(x)} g(\nabla_{\widetilde{v}X}\rho Y, \rho Z) + 2g(g(\nabla_{\widetilde{v}X}\rho Y, \overline{\zeta})\overline{\zeta}, \rho Z).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&e^{2\sigma(x)} g(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \rho Z) + g(g(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \overline{\zeta})\overline{\zeta}, \rho Z) \\
&\quad = e^{2\sigma(x)} g(\nabla_{\widetilde{v}X}\rho Y, \rho Z) + g(g(\nabla_{\widetilde{v}X}\rho Y, \overline{\zeta})\overline{\zeta}, \rho Z). \qquad (3.3)
\end{aligned}$$

From which, by setting $Z = \beta\overline{\zeta}$ and noting that $e^{2\sigma(x)} + p^2 \neq 0$, we get

$$g(\widetilde{\widetilde{\nabla}}_{\widetilde{v}X}\rho Y, \overline{\zeta}) - g(\nabla_{\widetilde{v}X}\rho Y, \overline{\zeta}) = 0. \tag{3.4}$$

Then, Eqs. (3.3) and (3.4) imply that

$$\widetilde{\widetilde{\nabla}}_{\widetilde{v}X}\rho Y = \nabla_{\widetilde{v}X}\rho Y. \tag{3.5}$$

Similarly, by Lemma 3.1(b), Proposition 2.5 and Lemma 3.2, noting that $T(\overline{\zeta}, \overline{X}) = T(\overline{X}, \overline{\zeta}) = 0$, we get after long but easy calculations,

$$\begin{aligned}
&2\widetilde{g}(\widetilde{\widetilde{\nabla}}_{\widetilde{h}X}\rho Y, \rho Z)\\
&= \widetilde{h}X \cdot \widetilde{g}(\rho Y, \rho Z) + \widetilde{h}Y \cdot \widetilde{g}(\rho Z, \rho X) - \widetilde{h}Z \cdot \widetilde{g}(\rho X, \rho Y)\\
&\quad - \widetilde{g}(\rho X, \rho[\widetilde{h}Y, \widetilde{h}Z]) + \widetilde{g}(\rho Y, \rho[\widetilde{h}Z, \widetilde{h}X]) + \widetilde{g}(\rho Z, \rho[\widetilde{h}X, \widetilde{h}Y])\\
&= (hX - \text{Ł}X) \cdot \left\{e^{2\sigma} g(\rho Y, \rho Z) + g(\rho Y, \overline{\zeta}) g(\rho Z, \overline{\zeta})\right\}\\
&\quad + (hY - \text{Ł}Y) \cdot \left\{e^{2\sigma} g(\rho Z, \rho X) + g(\rho Z, \overline{\zeta}) g(\rho X, \overline{\zeta})\right\}\\
&\quad - (hZ - \text{Ł}Z) \cdot \left\{e^{2\sigma} g(\rho X, \rho Y) + g(\rho X, \overline{\zeta}) g(\rho Y, \overline{\zeta})\right\}\\
&\quad - e^{2\sigma} g(\rho X, \rho[hY - \text{Ł}Y, hZ - \text{Ł}Z]) - g(\rho X, \overline{\zeta}) g(\rho[hY - \text{Ł}Y, hZ - \text{Ł}Z], \overline{\zeta})\\
&\quad + e^{2\sigma} g(\rho Y, \rho[hZ - \text{Ł}Z, hX - \text{Ł}X]) + g(\rho Y, \overline{\zeta}) g(\rho[hZ - \text{Ł}Z, hX - \text{Ł}X], \overline{\zeta})\\
&\quad + e^{2\sigma} g(\rho Z, \rho[vX + \text{Ł}X, hY - \text{Ł}Y]) + g(\rho Z, \overline{\zeta}) g(\rho[vX + \text{Ł}X, hY - \text{Ł}Y], \overline{\zeta}).\\
&= 2e^{2\sigma}\left\{g(\nabla_{hX}\rho Y, \rho Z) - g(\nabla_{\text{Ł}X}\rho Y, \rho Z)\right\} + 2e^{2\sigma}\{g(\rho X, \rho[\text{Ł}Y, hZ])\\
&\quad - g(\rho X, \nabla_{\text{Ł}Y}\rho Z)\} - 2e^{2\sigma}\{g(\rho Y, \rho[\text{Ł}Z, hX]) - g(\rho Y, \nabla_{\text{Ł}Z}\rho X)\}\\
&\quad + 2e^{2\sigma}\{g(d\sigma(X)\rho Y, \rho Z) + g(d\sigma(Y)\rho X, \rho Z) - g(\rho X, \rho Y) g(\overline{\sigma}, \rho Z)\}\\
&\quad + 2g(g(\nabla_{hX}\rho Y, \overline{\zeta})\,\overline{\zeta}, \rho Z) - 2g(g(\nabla_{\text{Ł}X}\rho Y, \overline{\zeta})\,\overline{\zeta}, \rho Z) - 2g(\rho Z, \overline{\zeta}) g(\rho X, \rho Y)\\
&= 2e^{2\sigma} g(\nabla_{\widetilde{h}X}\rho Y, \rho Z) + 2g(g(\nabla_{\widetilde{h}X}\rho Y, \overline{\zeta})\,\overline{\zeta}, \rho Z) - 2g(\rho Z, \overline{\zeta}) g(\rho X, \rho Y)\\
&\quad + 2e^{2\sigma} g(\rho Z, T'(\text{Ł}Y, hX)) - 2e^{2\sigma} g(\rho Z, \mathbf{T}(\text{Ł}Y, hX))\\
&\quad + 2e^{2\sigma}\left\{g(d\sigma(X)\rho Y, \rho Z) + g(d\sigma(Y)\rho X, \rho Z) - g(\rho X, \rho Y) g(\overline{\sigma}, \rho Z)\right\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&e^{2\sigma} g(\widetilde{\widetilde{\nabla}}_{\widetilde{h}X}\rho Y, \rho Z) + g(g(\widetilde{\widetilde{\nabla}}_{\widetilde{h}X}\rho Y, \overline{\zeta})\,\overline{\zeta}, \rho Z)\\
&\quad = e^{2\sigma} g(\nabla_{\widetilde{h}X}\rho Y, \rho Z) + g(g(\nabla_{\widetilde{h}X}\rho Y, \overline{\zeta})\overline{\zeta}, \rho Z) - g(\rho Z, \overline{\zeta}) g(\rho X, \rho Y)\\
&\quad\quad + e^{2\sigma} g(\rho Z, T'(\text{Ł}Y, hX)) - e^{2\sigma} g(\rho Z, \mathbf{T}(\text{Ł}Y, hX))\\
&\quad\quad + e^{2\sigma}\{g(d\sigma(X)\rho Y, \rho Z) + g(d\sigma(Y)\rho X, \rho Z) - g(\rho X, \rho Y) g(\overline{\sigma}, \rho Z)\}.
\end{aligned} \tag{3.6}$$

From which, setting $Z = \beta\overline{\zeta}$, we get

$$\widetilde{g}(\widetilde{\nabla}_{\widetilde{h}X}\rho Y, \overline{\zeta}) - g(\nabla_{\widetilde{h}X}\rho Y, \overline{\zeta}) = \frac{-p^2 g(\rho X, \rho Y) + e^{2\sigma}\left\{g(d\sigma(X)\rho Y, \rho Z) + g(d\sigma(Y)\rho X, \rho Z) - g(\rho X, \rho Y)g(\overline{\sigma}, \rho Z)\right\}}{e^{2\sigma} + p^2}. \tag{3.7}$$

Then, Eqs. (3.6) and (3.7) imply that

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{h}X}\rho Y &= \nabla_{\widetilde{h}X}\rho Y \\ &\quad - (d\sigma(hX))\rho Y + (d\sigma(hY))\rho X - g(\rho X, \rho Y)\overline{\sigma} - \mathbf{T}(ŁY, \rho X) + T'(ŁX, hY) \\ &\quad - \left\{\frac{d\sigma(hX)g(\rho Y, \overline{\zeta}) + d\sigma(hY)g(\rho X, \overline{\zeta}) - g(\overline{\sigma}, \overline{\zeta})g(\rho X, \rho Y) + g(\rho X, \rho Y)}{e^{2\sigma(x)} + p^2}\right\}\overline{\zeta},\end{aligned} \tag{3.8}$$

Now, (3.1) follows from (3.5) and (3.8). □

Remark 3.4. From Theorem 3.3(a), we conclude that the tensor field $A(\overline{X}, \overline{Y}) := \omega(\gamma\overline{X}, \overline{Y})$ vanishes.

As a consequence of Theorem 3.3 and Definition 2.1, we have

Corollary 3.5. *If a Finsler manifold admits a concurrent π-vector field $\overline{\zeta}$, then the vector field $\overline{\zeta}$ is no more concurrent with respect to the transformed metric* (2.1)*.*

Corollary 3.6. *Under the energy β-conformal change* (2.1), *we have*:

(a) *The maps $\overline{Y} \mapsto \nabla_{\gamma\overline{X}}\overline{Y}$ and $\overline{Y} \mapsto \nabla_{\gamma\overline{Y}}\overline{X}$ are invariant.*
(b) *The (h)hv-torsion T is invariant.*

We retrieve here a result of [17, 18]:

Corollary 3.7.

(a) *Let the energy β-conformal change* (2.1) *be a conformal change* ($\zeta = 0$), *then we have*

$$\begin{aligned}\widetilde{\nabla}_X\overline{Y} &= \nabla_X\overline{Y} + (d\sigma(hX))\overline{Y} + (d\sigma(\beta\overline{Y}))\rho X - g(\rho X, \overline{Y})\overline{\sigma} \\ &\quad - \boldsymbol{T}(Ł_1\beta\overline{Y}, \rho X) + T'(Ł_1 X, \beta\overline{Y}),\end{aligned}$$

where $Ł_1$ is given by Lemma 3.2 *and T' is given by Theorem* 3.3*.*

(b) *Let the energy β-conformal change* (2.1) *be an energy β-change* ($\sigma = 0$), *then we have*

$$\widetilde{\nabla}_X\rho Y = \nabla_X\rho Y - \frac{g(\rho X, \rho Y)}{1 + p^2}\overline{\zeta},$$

where $p^2 := g(\overline{\zeta}, \overline{\zeta})$ and $\overline{\zeta}$ is a concurrent π-vector field.

Theorem 3.8. *Let (M,L) and $(M,\widetilde{L})$ be two Finsler manifolds related by the energy β-conformal change* (2.1). *The curvature tensors of the associated Cartan connections ∇ and $\widetilde{\nabla}$ are related by*:

(a) $\widetilde{S}(\overline{X},\overline{Y})\overline{Z}=S(\overline{X},\overline{Y})\overline{Z}$,
(b) $\widetilde{P}(\overline{X},\overline{Y})\overline{Z}=P(\overline{X},\overline{Y})\overline{Z}+V(\overline{X},\overline{Y})\overline{Z}$,

where V is the vector π-form defined by

$$V(\overline{X},\overline{Y})\overline{Z}=-S(N\overline{X},\overline{Y})\overline{Z}+(\nabla_{\gamma\overline{Y}}B)(\overline{X},\overline{Z})+B(T(\overline{Y},\overline{X}),\overline{Z}) \tag{3.9}$$

(c) $\widetilde{R}(\overline{X},\overline{Y})\overline{Z}=R(\overline{X},\overline{Y})\overline{Z}+H(\overline{X},\overline{Y})\overline{Z}$,
where H is the vector π-form defined by

$$\begin{aligned}H(\overline{X},\overline{Y})\overline{Z}&=S(N\overline{X},N\overline{Y})\overline{Z}-\mathfrak{A}_{\overline{X},\overline{Y}}\{P(\overline{X},N\overline{Y})\overline{Z}+(\nabla_{\beta\overline{X}}B)(\overline{Y},\overline{Z})\\&\quad-(\nabla_{\gamma N\overline{X}}B)(\overline{Y},\overline{Z})+B(\overline{X},B(\overline{Y},\overline{Z}))-B(T(N\overline{X},\overline{Y}),\overline{Z})\},\end{aligned} \tag{3.10}$$

B is defined in Theorem 3.3 *and $L:=\gamma\circ N\circ\rho$ is given by Lemma* 3.2.

Proof. We have

$$\begin{aligned}\widetilde{D}_X\widetilde{\nabla}_Y\overline{Z}&=\widetilde{\nabla}_X(\nabla_Y\overline{Z}+\omega(Y,\overline{Z}))\\&=\nabla_X\nabla_Y\overline{Z}+\omega(X,\nabla_Y\overline{Z})+\nabla_X\omega(Y,\overline{Z})+\omega(X,\omega(Y,\overline{Z}))\\&=\nabla_X\nabla_Y\overline{Z}+B(\rho X,\nabla_Y\overline{Z})+A(KX,\nabla_Y\overline{Z})+\nabla_XB(\rho Y,\overline{Z})\\&\quad+\nabla_XA(KY,\overline{Z})+A(KX,A(KY,\overline{Z}))+A(KX,B(\rho Y,\overline{Z}))\\&\quad+B(\rho X,A(KY,\overline{Z}))+B(\rho X,B(\rho Y,\overline{Z})).\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{\nabla}_Y\widetilde{\nabla}_X\overline{Z}&=\nabla_Y\nabla_X\overline{Z}+B(\rho Y,\nabla_X\overline{Z})+A(KY,\nabla_X\overline{Z})+\nabla_YB(\rho X,\overline{Z})\\&\quad+\nabla_YA(KX,\overline{Z})+A(KY,A(KX,\overline{Z}))+A(KY,B(\rho X,\overline{Z}))\\&\quad+B(\rho Y,A(KX,\overline{Z}))+B(\rho Y,B(\rho X,\overline{Z})).\end{aligned}$$

Moreover,

$$\begin{aligned}\widetilde{\nabla}_{[X,Y]}\overline{Z}&=\nabla_{[X,Y]}\overline{Z}+\omega([X,Y],\overline{Z})\\&=\nabla_{[X,Y]}\overline{Z}+B(\rho[X,Y],\overline{Z})+A(K[X,Y],\overline{Z})\\&=\nabla_{[X,Y]}\overline{Z}+B(\nabla_X\rho Y-\nabla_Y\rho X-\mathbf{T}(X,Y),\overline{Z})+A(\mathbf{K}(X,Y)\overline{\eta},\overline{Z})\\&\quad+A(\nabla_XKY-\nabla_YKX,\overline{Z}).\end{aligned}$$

From the above relation and the definition of classical curvature of ∇, we get

$$\begin{aligned}\widetilde{\mathbf{K}}(X,Y)\overline{Z}\\ &= \mathbf{K}(X,Y)\overline{Z} + A(\mathbf{K}(X,Y)\overline{\eta},\overline{Z}) - B(\mathbf{T}(X,Y),\overline{Z}) - \mathfrak{U}_{X,Y}\{(\nabla_X A)(KY,\overline{Z})\\ &\quad + (\nabla_X B)(\rho Y,\overline{Z}) + A(KX, A(KY,\overline{Z})) + A(KX, B(\rho Y,\overline{Z}))\\ &\quad + B(\rho X, A(KY,\overline{Z})) + B(\rho X, B(\rho Y,\overline{Z}))\}. \end{aligned} \tag{3.11}$$

Now, (a) follows from (3.11) by setting $X = \gamma\overline{X}$ and $Y = \gamma\overline{Y}$, noting that $\rho\, o\, \gamma = 0$ and $A = 0$ (Remark 3.4), whereas (b) follows from (3.11) by setting $X = \widetilde{\beta}\overline{X}$ and $Y = \gamma\overline{Y}$, taking into account the identity $A = 0$. Finally, (c) follows from the same relation by setting $X = \widetilde{\beta}\overline{X}$ and $Y = \widetilde{\beta}\overline{Y}$, noting that $\widetilde{\beta} = \beta - \gamma \circ N$ ([17]) and the fact that $A = 0$. □

In view of the above theorem, we have the following corollary.

Corollary 3.9. *The v-curvature tensor S is invariant under the energy β-conformal change* (2.1).

We here turn our attention to the energy β-conformal change of Berwald connection.

Theorem 3.10. *Let (M, L) and $(M, \widetilde{L})$ be two Finsler manifolds related by the energy β-conformal change* (2.1). *Then the associated Berwald connections D° and $\widetilde{D^\circ}$ are related by:*

$$\widetilde{D^\circ}_X\overline{Y} = D^\circ_X\overline{Y} + \omega^\circ(X,\overline{Y}), \tag{3.12}$$

where $\omega^\circ(X,\overline{Y}) = (D^\circ_{\gamma\overline{Y}}N)(\rho X)$.

In particular, we have

(a) $\widetilde{D^\circ}_{\gamma\overline{X}}\overline{Y} = D^\circ_{\gamma\overline{X}}\overline{Y}$,

(b) $\widetilde{D^\circ}_{\widetilde{\beta}\overline{X}}\overline{Y} = D^\circ_{\beta\overline{X}}\overline{Y} + B^\circ(\overline{X},\overline{Y}) - D^\circ_{\gamma N\overline{X}}\overline{Y}$,

where $B^\circ(\overline{X},\overline{Y}) := \omega^\circ(\beta\overline{X},\overline{Y})$.

Proof. Formula (3.12) follows from Theorems 1.3, 3.3, 3.8 and Lemma 3.2. In more detail,

$$\begin{aligned}\widetilde{D^\circ}_X\overline{Y} &= \widetilde{\nabla}_X\overline{Y} + \widetilde{P}(\rho X,\overline{Y})\overline{\eta} - \widetilde{T}(\widetilde{K}X,\overline{Y})\\ &= \nabla_X\overline{Y} + B(\rho X,\overline{Y}) + P(\rho X,\overline{Y})\overline{\eta} + V(\rho X,\overline{Y})\overline{\eta} - T(KX,\overline{Y}) - T(K\L X,\overline{Y})\\ &= D^\circ_X\overline{Y} + B(\rho X,\overline{Y}) + \nabla_{\gamma\overline{Y}}B(\rho X,\overline{\eta}) - B(\nabla_{\gamma\overline{Y}}\rho X,\overline{\eta})\\ &\quad - B(\rho X,\overline{Y}) + B(T(\overline{Y},\rho X),\overline{\eta}) - T(N\rho X,\overline{Y})\\ &= D^\circ_X\overline{Y} + (D^\circ_{\gamma\overline{Y}}N)(\rho X) = D^\circ_X\overline{Y} + \omega^\circ(X,\overline{Y}).\end{aligned}$$

Relations (a) and (b) follow from (3.12) by setting $X = \gamma\overline{X}$ and $X = \widetilde{\beta}\overline{X}$, respectively. □

Remark 3.11. From Theorem 3.10(a), we conclude that the tensor field $A^\circ(\overline{X},\overline{Y}) := \omega^\circ(\gamma\overline{X},\overline{Y})$ vanishes.

Theorem 3.12. *Under the energy β-conformal change* (2.1), *we have*:

(a) $\widetilde{S^\circ}(\overline{X},\overline{Y})\overline{Z} = S^\circ(\overline{X},\overline{Y})\overline{Z} = 0,$
(b) $\widetilde{P^\circ}(\overline{X},\overline{Y})\overline{Z} = P^\circ(\overline{X},\overline{Y})\overline{Z} + (D^\circ_{\gamma\overline{Y}}B^\circ)(\overline{X},\overline{Z}),$
(c) $$\begin{aligned}\widetilde{R^\circ}(\overline{X},\overline{Y})\overline{Z} &= R^\circ(\overline{X},\overline{Y})\overline{Z} + \mathfrak{U}_{\overline{X},\overline{Y}}\{P^\circ(\overline{Y},N\overline{X})\overline{Z} - (D^\circ_{\beta\overline{X}}B^\circ)(\overline{Y},\overline{Z})\\ &\quad + (D^\circ_{\gamma N\overline{X}}B^\circ)(\overline{Y},\overline{Z}) - B^\circ(\overline{X},B^\circ(\overline{Y},\overline{Z}))\}\\ &=: R^\circ(\overline{X},\overline{Y})\overline{Z} + H^\circ(\overline{X},\overline{Y})\overline{Z},\end{aligned}$$

where $B^\circ(\overline{X},\overline{Y}) := \omega^\circ(\beta\overline{X},\overline{Y})$.

Proof. The proof is similar to that of Theorem 3.8, taking into account the properties of Berwald connection together with Remark 3.11. □

Now, we investigate the effect of the energy β-conformal change (2.1) on the Chern connection D^c.

Theorem 3.13 ([19]). *Let (M,L) be a Finsler manifold. The Chern connection D^c is expressed in terms of the Cartan connection as*

$$D^c_X\overline{Y} = \nabla_X\overline{Y} - T(KX,\overline{Y}).$$

In particular, we have:

(a) $D^c_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X},\overline{Y}),$
(b) $D^c_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y}.$

Theorem 3.14. *Let (M,L) and $(M,\widetilde{L})$ be two Finsler manifolds related by the energy β-conformal change* (2.1). *Then, the associated Chern connections D^c and $\widetilde{D^c}$ are related by*

$$\widetilde{D^c}_X\overline{Y} = D^c_X\overline{Y} + \omega^c(X,\overline{Y}), \tag{3.13}$$

where

$$\begin{aligned}&\omega^c(X,\overline{Y})\\ &\quad := \nabla_{\widetilde{h}X}\rho Y - (d\sigma(hX))\rho Y + (d\sigma(hY))\rho X - g(\rho X,\rho Y)\overline{\sigma}\\ &\qquad - T(N\overline{Y},\rho X) + T'(ŁX,\beta\overline{Y}) - T(N\rho X,\overline{Y})\\ &\qquad - \left\{\frac{d\sigma(hX)g(\rho Y,\overline{\zeta}) + d\sigma(hY)g(\rho X,\overline{\zeta}) - g(\overline{\sigma},\overline{\zeta})g(\rho X,\rho Y) + g(\rho X,\rho Y)}{e^{2\sigma(x)}+p^2}\right\}\overline{\zeta},\end{aligned}$$

In particular, we have:

(a) $\widetilde{D^c}_{\gamma\overline{X}}\overline{Y} = D^c_{\gamma\overline{X}}\overline{Y},$
(b) $\widetilde{D^c}_{\widetilde{\beta}\overline{X}}\overline{Y} = D^c_{\beta\overline{X}}\overline{Y} + B^c(\overline{X},\overline{Y}) - D^c_{Ł\beta\overline{X}}\overline{Y},$

where $B^c(\overline{X},\overline{Y}) := \omega^c(\beta\overline{X},\overline{Y})$.

Proof. Formula (3.13) follows from Theorems 3.13, 3.3 and Lemma 3.2, taking into account the fact that the (h)hv-torsion tensor T is invariant (Corollary 3.6(b)):

$$\begin{aligned}\widetilde{D^c}_X\overline{Y} &= \widetilde{\nabla}_X\overline{Y} - \widetilde{T}(\widetilde{K}X,\overline{Y})\\ &= \nabla_X\overline{Y} + \omega(X,\overline{Y}) - T(KX,\overline{Y}) - T(KŁX,\overline{Y})\\ &= D^c_X\overline{Y} + \omega^c(X,\overline{Y}).\end{aligned}$$

Relations (a) and (b) follow from (3.13) by setting $X = \gamma\overline{X}$ and $X = \widetilde{\beta}\overline{X}$ respectively. □

Remark 3.15. From Theorem 3.14(a), we conclude that the tensor field $A^c(\overline{X},\overline{Y}) := \omega^c(\gamma\overline{X},\overline{Y})$ vanishes.

In view of the above theorem, we have the following theorem.

Theorem 3.16. *Under the energy β-conformal change* (2.1), *we have*

(a) $\widetilde{S^c}(\overline{X},\overline{Y})\overline{Z} = S^c(\overline{X},\overline{Y})\overline{Z} = 0$,
(b) $\widetilde{P^c}(\overline{X},\overline{Y})\overline{Z} = P^c(\overline{X},\overline{Y})\overline{Z} + (D^c_{\gamma\overline{Y}}B^c)(\overline{X},\overline{Z})$,
(c) $\widetilde{R^c}(\overline{X},\overline{Y})\overline{Z} = R^c(\overline{X},\overline{Y})\overline{Z} + \mathfrak{U}_{\overline{X},\overline{Y}}\{P^c(\overline{Y},N\overline{X})\overline{Z} - (D^c_{\beta\overline{X}}B^c)(\overline{Y},\overline{Z}) + (D^c_{\gamma N\overline{X}}B^c)(\overline{Y},\overline{Z}) - B^c(\overline{X},B^c(\overline{Y},\overline{Z}))\}$.

Proof. The proof is similar to that of Theorem 3.8, taking into account the properties of Chern connection together with Remark 3.15. □

Finally, we investigate the effect of the energy β-conformal change (2.1) on the Hashiguchi connection D^*.

Theorem 3.17 ([19]). *Let* (M,L) *be a Finsler manifold. The Hashiguchi connection* D^* *is expressed in terms of the Cartan connection* ∇ *as*

$$D^*_X\overline{Y} = \nabla_X\overline{Y} + \widehat{P}(\rho X,\overline{Y}).$$

In particular, we have:

(a) $D^*_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y}$,
(b) $D^*_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X},\overline{Y})$.

Theorem 3.18. *Let* (M,L) *and* $(M,\widetilde{L})$ *be two Finsler manifolds related by the energy β-conformal change* (2.1). *Then, the associated Hashiguchi connections* D^* *and* $\widetilde{D^*}$ *are related by*

$$\widetilde{D^*}_X\overline{Y} = D^*_X\overline{Y} + \omega^*(X,\overline{Y}), \tag{3.14}$$

where $\omega^*(X,\overline{Y}) = (D^*_{\gamma\overline{Y}}N)(\rho X) + N(T(\overline{Y},\rho X))$.

In particular, we have

(a) $\widetilde{D^*}_{\gamma\overline{X}}\overline{Y} = D^*_{\gamma\overline{X}}\overline{Y}$,
(b) $\widetilde{D^*}_{\widetilde{\beta}\overline{X}}\overline{Y} = D^*_{\beta\overline{X}}\overline{Y} + B^*(\overline{X},\overline{Y}) - D^*_{\gamma N\overline{X}}\overline{Y}$,

where $B^*(\overline{X},\overline{Y}) := \omega^*(\beta\overline{X},\overline{Y})$.

Proof. Formula (3.14) follows from Theorems 3.17, 3.3, 3.8 and Lemma 3.2. In more details,

$$\begin{aligned}\widetilde{D^*}_X\overline{Y} &= \widetilde{\nabla}_X\overline{Y} + \widetilde{P}(\rho X,\overline{Y})\overline{\eta}\\ &= \nabla_X\overline{Y} + B(\rho X,\overline{Y}) + P(\rho X,\overline{Y})\overline{\eta} + V(\rho X,\overline{Y})\overline{\eta}\\ &= D^*_X\overline{Y} + B(\rho X,\overline{Y}) + \nabla_{\gamma\overline{Y}}B(\rho X,\overline{\eta}) - B(\nabla_{\gamma\overline{Y}}\rho X,\overline{\eta})\\ &\quad - B(\rho X,\overline{Y}) + B(T(\overline{Y},\rho X),\overline{\eta})\\ &= D^*_X\overline{Y} + (\nabla_{\gamma\overline{Y}}N)(\rho X) + N(T(\overline{Y},\rho X))\\ &= D^*_X\overline{Y} + (D^*_{\gamma\overline{Y}}N)(\rho X) + N(T(\overline{Y},\rho X))\\ &= D^*_X\overline{Y} + \omega^*(X,\overline{Y}).\end{aligned}$$

Relation (a) follows from (3.14) by setting $X = \gamma\overline{X}$ noting that $\rho\circ\gamma = 0$, whereas, relation (b) follows from the same formula by setting $X = \widetilde{\beta}\overline{X}$, noting that $\widetilde{\beta} = \beta - Ł\circ\beta$. □

In view of the above theorem, we have the following theorem.

Theorem 3.19. *Under the energy β-conformal change* (2.1), *we have*

(a) $\widetilde{S^*}(\overline{X},\overline{Y})\overline{Z} = S^*(\overline{X},\overline{Y})\overline{Z}$.
(b) $\widetilde{P^*}(\overline{X},\overline{Y})\overline{Z} = P^*(\overline{X},\overline{Y})\overline{Z} - S(N\overline{X},\overline{Y})\overline{Z} + (D^*_{\gamma\overline{Y}}B^*)(\overline{X},\overline{Z}) + B^*(T(\overline{Y},\overline{X}),\overline{Z})$.
(c) $$\begin{aligned}\widetilde{R^*}(\overline{X},\overline{Y})\overline{Z} = R^*(\overline{X},\overline{Y})\overline{Z} &+ S(N\overline{X},N\overline{Y})\overline{Z}\\ &- \mathfrak{U}_{\overline{X},\overline{Y}}\{P^*(\overline{X},N\overline{Y})\overline{Z} + (D^*_{\beta\overline{X}}B^*)(\overline{Y},\overline{Z})\\ &- (D^*_{\gamma N\overline{X}}B^*)(\overline{Y},\overline{Z}) + B^*(\overline{X},B^*(\overline{Y},\overline{Z})) - B^*(T(N\overline{X},\overline{Y}),\overline{Z})\}.\end{aligned}$$

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